Robustness of continuous-variable entanglement under signal amplification and attenuation

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entanglement
Alice


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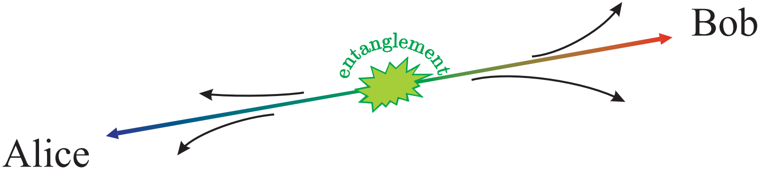
entanglement
Alice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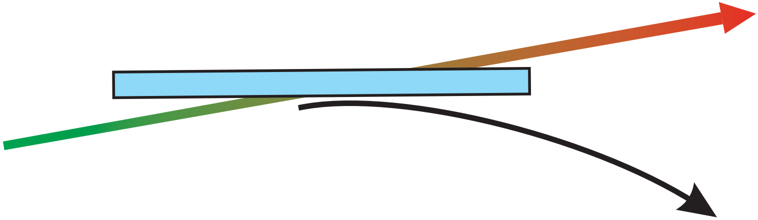

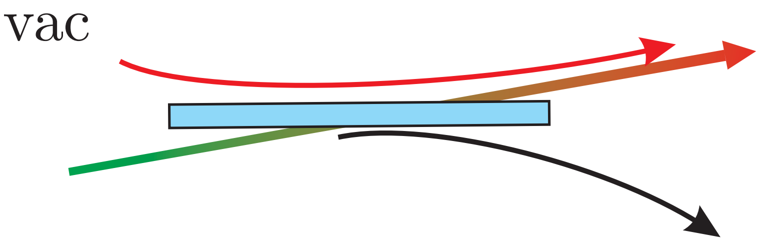
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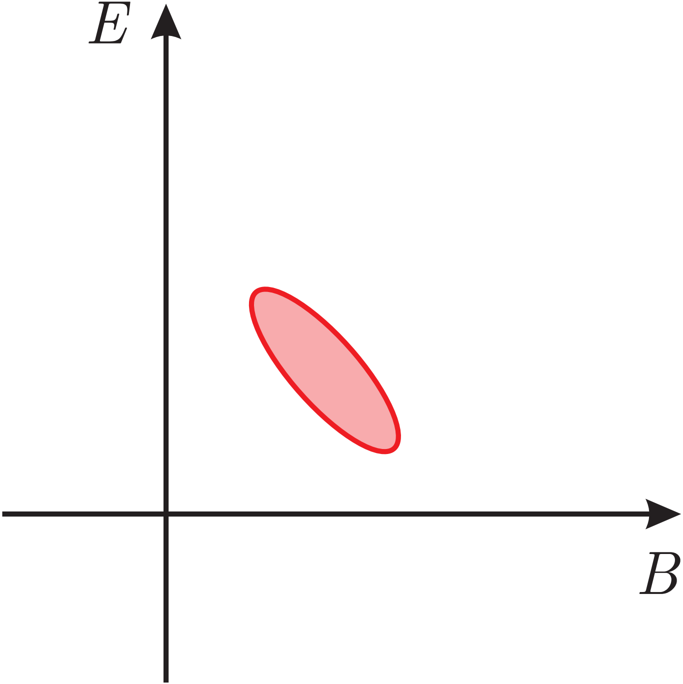
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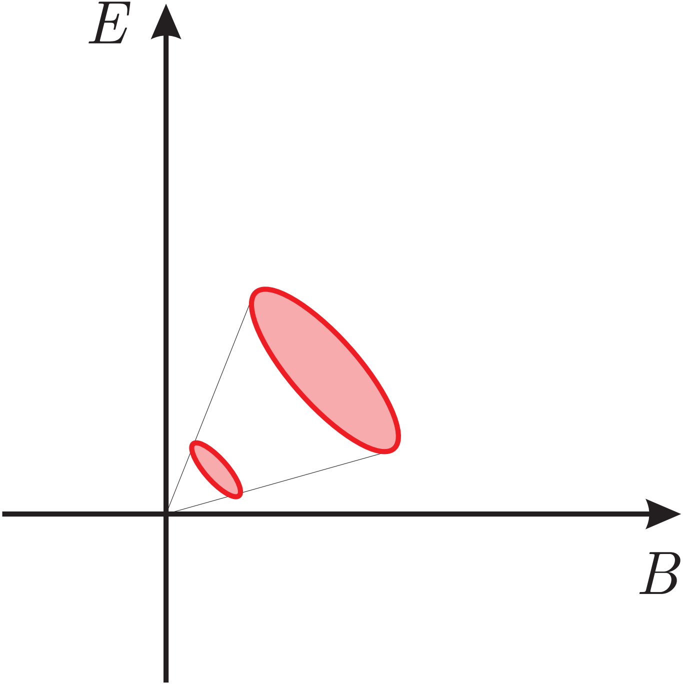

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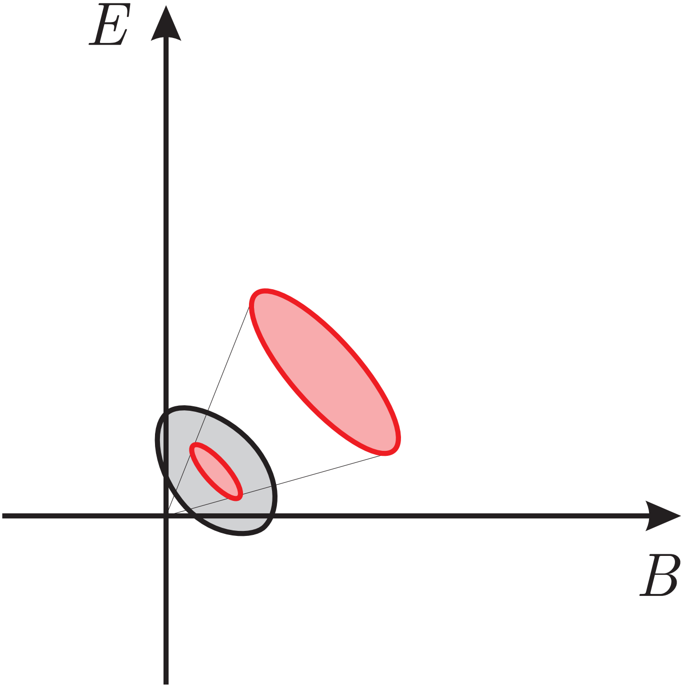
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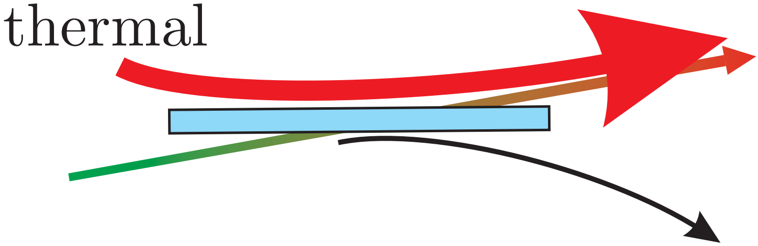
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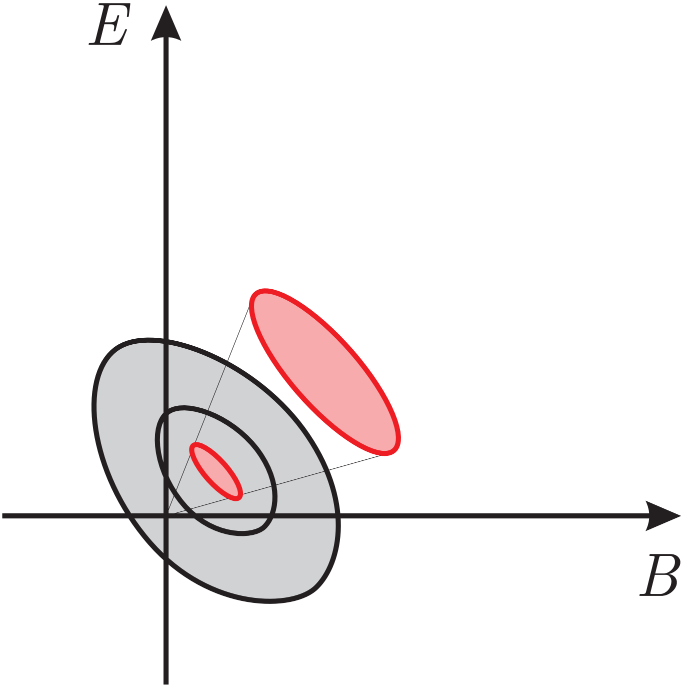

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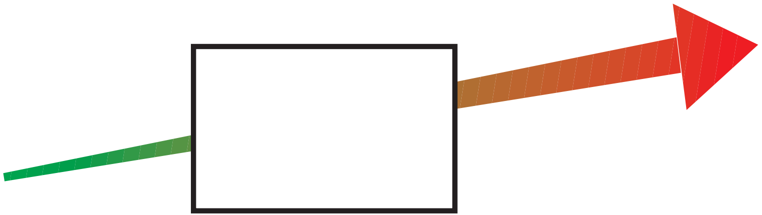
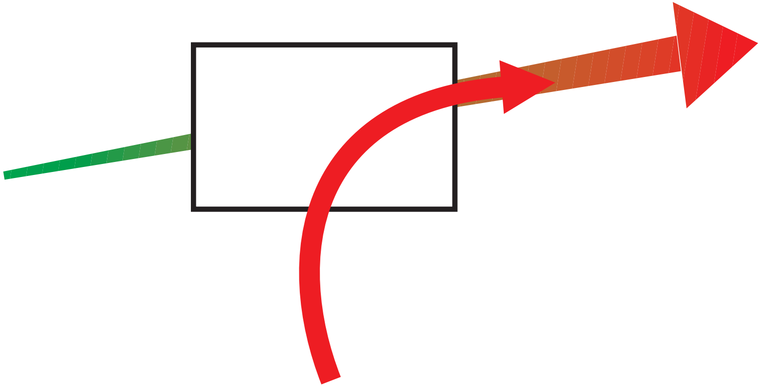
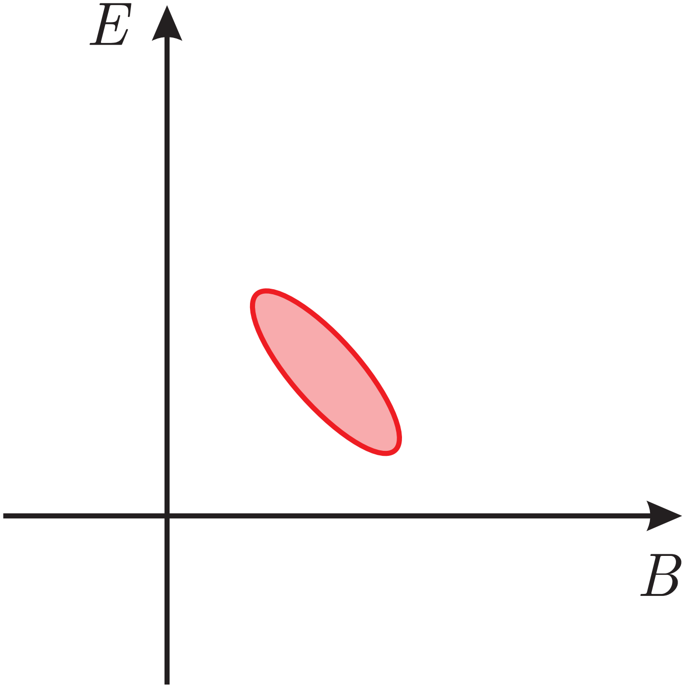
TO REMEMBER

Losses $\Rightarrow$ noise



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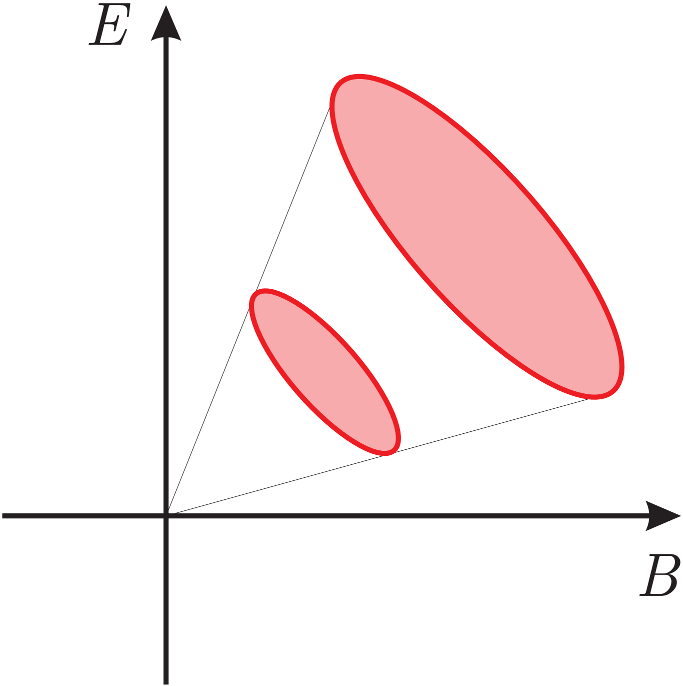
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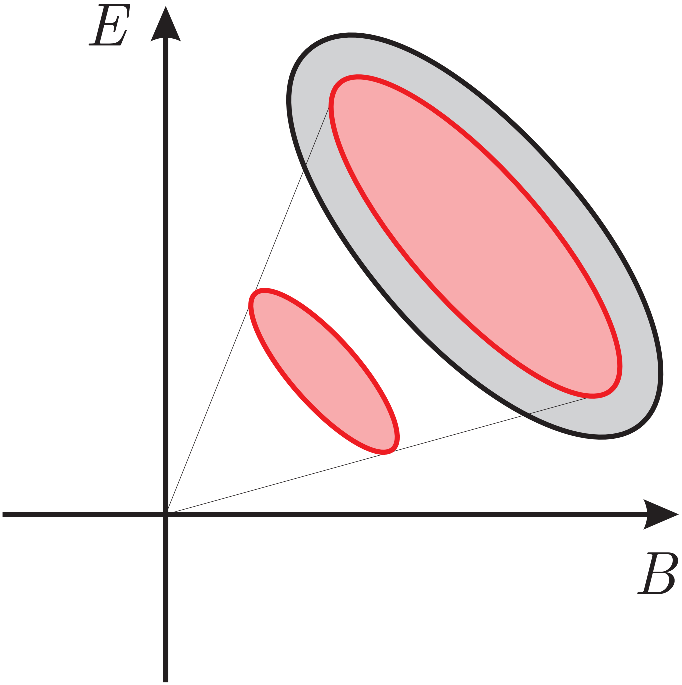
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TO REMEMBER

Amplification $\Rightarrow$ noise

N-mode states

Characteristic function $\varphi(\mathbf{z}) = \mathrm{tr}[\varrho W(\mathbf{z})]$

- coordinates in the real symplectic space
  $\mathbf{z} = (x_1, y_1, \ldots, x_N, y_N)^\top$
- Weyl operator
  $W(\mathbf{z}) = \exp[i(q_1 x_1 + p_1 y_1 + \cdots + q_N x_N + p_N y_N)]$
- canonical commutation relation $\quad [q_i, p_j] = i\delta_{ij}$
- number of modes $\quad N$

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Gaussian states: $\varphi(\mathbf{z}) = \exp\left(-\frac{1}{2}\mathbf{z}^\top \mathbf{V}\mathbf{z} + i\mathbf{l}^\top \mathbf{z}\right)$

- $\mathbf{V}$ is covariance matrix satisfying $\mathbf{V} \geqslant \frac{i}{2}\mathbf{\Delta}$
- symplectic form $\quad \mathbf{\Delta} = \bigoplus_{i=1}^{N} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Examples of Gaussian states

Examples of Gaussian states

- 1-mode coherent state $|\gamma\rangle$: $\frac{1}{\sqrt{2}}(q+ip)|\gamma\rangle = \gamma|\gamma\rangle$, $\gamma \in \mathbb{C}$
  $|\gamma\rangle = \exp(-\frac{|\gamma|^2}{2}) \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{n!}}|n\rangle$, $\frac{1}{2}(q^2+p^2)|n\rangle = (n+\frac{1}{2})|n\rangle$

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- 2-mode squeezed vacuum
  $|\psi\rangle = \sqrt{1-(\tanh r)^2}\sum_{n=0}^{\infty}(\tanh r)^n|n\rangle \otimes |n\rangle$

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{pmatrix}.$$

Attenuator and amplifier

$$\varphi_{\text{out}}(\mathbf{z}) = \varphi_{\text{in}}(\mathbf{K}\mathbf{z}) \exp\left(-\frac{1}{2}\mathbf{z}^\top \mathbf{M}\mathbf{z}\right)$$

$$\mathbf{K} = \sqrt{\kappa}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{M} = \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

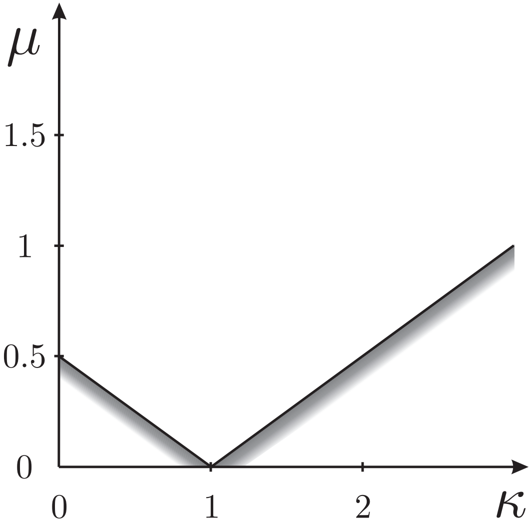
- attenuation $(0 < \kappa < 1)$
- addition of classical noise $(\kappa = 1)$
- amplification $(\kappa > 1)$

Holevo, Werner'01:
Fair physical channels (completely positive maps): $\mu \geq \frac{1}{2}|\kappa - 1|$

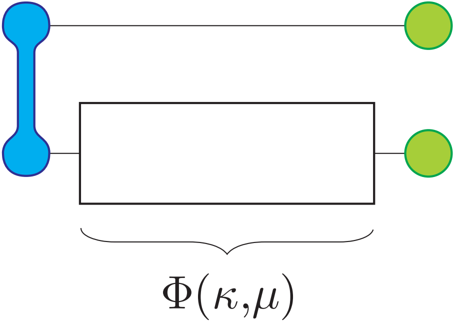
Quantum limited operation: $\mu_{\text{QL}} = \frac{1}{2}|\kappa - 1|$

Extra noise: $a = \mu - \mu_{\text{QL}} \geq 0$

Attenuator and amplifier



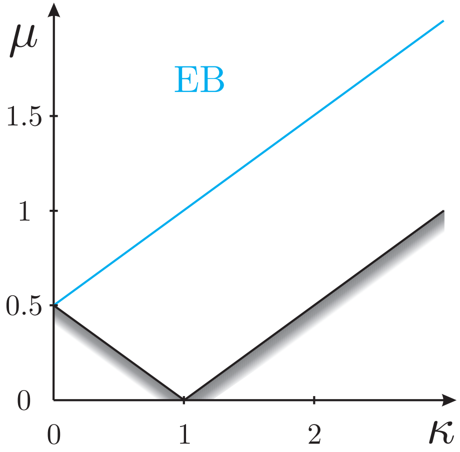
One-sided disentangling noise



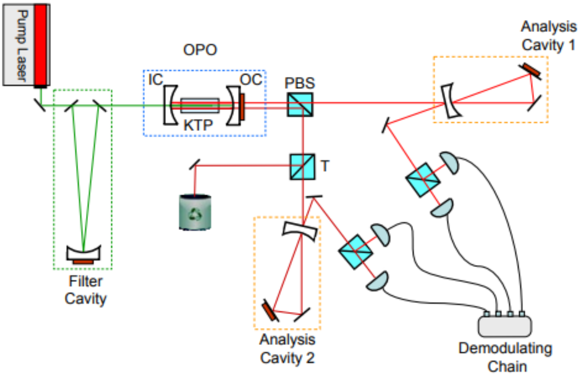
Definition

Channel Φ is entanglement breaking (EB) if for an arbitrary input state of the channel $\Phi \otimes \mathrm{Id}$ its output state is separable.

Attenuator and amplifier

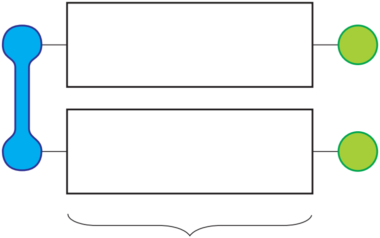


Holevo'08: $\Phi(\kappa, \mu)$ is EB iff $\mu \geqslant \frac{1}{2}(1 + \kappa)$.



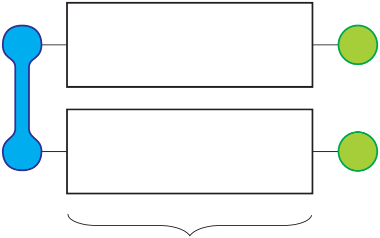
F. A. S. Barbosa, A. S. Coelho, A. J. de Faria, K. N. Cassemiro, A. S. Villar, P. Nussenzveig, M. Martinelli. Robustness of bipartite Gaussian entangled beams propagating in lossy channels. Nature Photonics **4**, 858 (2010)

N-sided disentangling noise



$\Phi(\kappa,\mu)$

N-sided disentangling noise



$\Phi(\kappa,\mu)$

Definition

Channel Φ is N-locally entanglement annihilating (N-LEA) if for an arbitrary input state of the channel $\Phi^{\otimes N}$ its output state is separable. (Moravčíková, Ziman'10)

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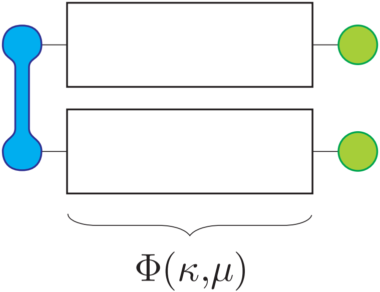

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Definition

Channel Φ is N-LEA$_G$ if for an arbitrary Gaussian input state of the channel $\Phi^{\otimes N}$ its output state is separable.

N-LEA$_{\mathcal{G}}$ attenuators and amplifiers

Proposition (SF, Ziman'14). The channel $\Phi(\kappa, \mu)$ is N-LEA$_{\mathcal{G}}$ if and only if the total noise level $\mu \geq \frac{1}{2}$.

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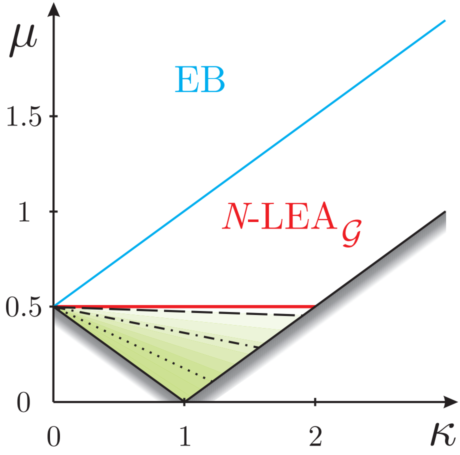

Scaling map

O. Man'ko, V. Man'ko, Marmo, Shaji, Sudarshan, Zaccaria'05

$$\Xi : \qquad \varphi_{\mathrm{out}}(\mathbf{z}) = \varphi_{\mathrm{in}}(\sqrt{\kappa_\Xi}\,\mathbf{z})$$

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Lemma

Scaling map Ξ transforms Gaussian states into Gaussian states if $\kappa_\Xi \geqslant 1$.

Proof. $\mathbf{V}_{\text{out}} = \kappa_\Xi \mathbf{V}_{\text{in}} \geqslant \mathbf{V}_{\text{in}} \geqslant \frac{i}{2}\mathbf{\Delta}$.

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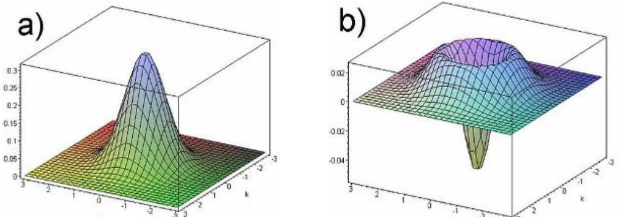
Lemma

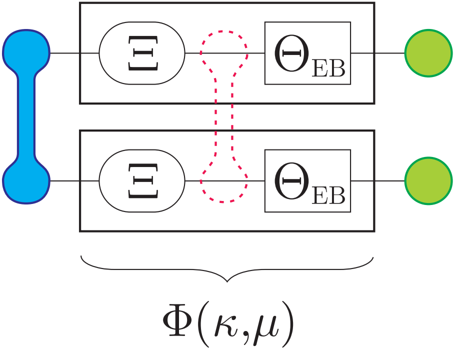
Scaling map Ξ is not positive.

Proof. $|\langle 0|2\rangle|^2 = 0$ but $\langle 0|\,\Xi[|2\rangle\langle 2|]\,|0\rangle < 0$.



Proof.

Sufficiency.



The relation $\Phi(\kappa,\mu) = \Theta_{\mathrm{EB}} \circ \Xi$ holds if $\kappa = \kappa_\Xi \kappa_\Theta$ and $\mu = \mu_\Theta \geq \frac{1}{2}(1+\kappa_\Theta)$.

In the limit $\kappa_\Theta \to 0$ and $\kappa_\Xi \to \infty$ with keeping $\kappa_\Xi \kappa_\Theta = \kappa = \mathrm{const}$, we obtain $\mu \geq \frac{1}{2}$.

Necessity.
If $\mu < \frac{1}{2}$ then $\Phi(\kappa, \mu)^{\otimes 2}$ preserves entanglement of the 2-mode squeezed vacuum state $|\psi\rangle = \sqrt{1 - \tanh^2 r} \sum_{n=0}^{\infty} (\tanh r)^n |n\rangle \otimes |n\rangle$ when $r \to \infty$. This can be checked, e.g., by Simon's criterion applied to the covariance matrix

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{pmatrix}.$$

Thus, $\Phi(\kappa, \mu)$ is not 2-LEA$_{\mathcal{G}}$ and, consequently, not N-LEA$_{\mathcal{G}}$.

Corollary

The channel $\Upsilon = \Phi(\kappa_1, \mu_1) \otimes \Phi(\kappa_2, \mu_2) \otimes \cdots \otimes \Phi(\kappa_N, \mu_N)$ *annihilates entanglement of all gaussian* N*-mode states if* $\mu_i \geq \frac{1}{2}$*,* $i = 1, \ldots, N$*.*

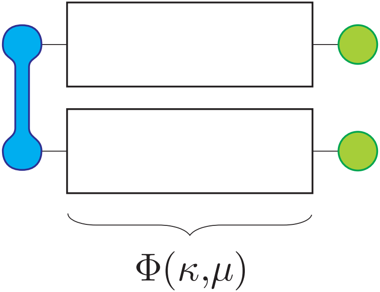
Proof. Similarly to the proof of Proposition, concatenation of the homogeneous scaling map $\Xi^{\otimes N}$ and a map $\bigotimes_{i=1}^{N} \Theta_{\mathrm{EB}i}$ composed of the individual entanglement-breaking attenuators with $a_{\Theta i} = \kappa_{\Theta i} = \kappa_i / \kappa_\Xi$ leads to the map $\Phi(\kappa_1, \frac{1}{2}) \otimes \Phi(\kappa_2, \frac{1}{2}) \otimes \cdots \otimes \Phi(\kappa_N, \frac{1}{2})$ in the limit $\kappa_\Xi \to \infty$. The map Υ may be readily obtained by adding classical noise $(\mu_i - \frac{1}{2})$ into ith mode, $i = 1, \ldots, N$.

Corollary

Suppose the channel $\Upsilon = \Phi(\kappa_1, \mu_1) \otimes \Phi(\kappa_2, \mu_2) \otimes \cdots \otimes \Phi(\kappa_N, \mu_N)$ *such that* $\min_{i=1,\dots,N-1} \frac{2\mu_i - 1}{\kappa_i} = s > 0$*. The channel* Υ *annihilates entanglement of all gaussian* N*-mode states if* $\mu_N \geq \frac{1}{2}(1 - s\kappa_N)$*.*

Proof. If $s \geq 1$, then all the channels $\Phi(\kappa_1, \mu_1)$, $\Phi(\kappa_2, \mu_2)$, ..., $\Phi(\kappa_{N-1}, \mu_{N-1})$ are EB and the statement becomes trivial. If $0 < s < 1$, let us represent Υ as a concatenation of the homogeneous scaling map $\Xi^{\otimes N}$ and a map $\left(\bigotimes_{i=1}^{N-1} \Theta_{\mathrm{EB}i}\right) \otimes \Theta_{\mathrm{QL}N}$ composed of $N-1$ entanglement breaking attenuators and a quantum limited attenuation of the Nth mode. In fact, put $\kappa_\Xi = s^{-1} > 1$ and $\kappa_{\Theta i} = s\kappa_i$, $i = 1, \dots, N-1$, then the relation $a_{\Theta i} = \mu_i - \frac{1}{2}(1 - s\kappa_i) \geq \kappa_{\Theta i}$ makes Θ_i entanglement breaking for $i = 1, \dots, N-1$, which guarantees separability of the output state for all N-mode gaussian inputs. The application of the quantum limited attenuator $\Theta_{\mathrm{QL}N}$ with $\kappa_{\Theta N} = s\kappa_N$ results in the noise $\mu_N = \frac{1}{2}(1 - \kappa_{\Theta N}) = \frac{1}{2}(1 - s\kappa_N)$. Greater noises in Nth mode can be realized by adding classical noise.

Non-Gaussian input states



What happens if we do not restrict to Gaussian inputs?

Proposition (SF, Ziman'14)

The channel $\Phi(\kappa_1, \mu_1) \otimes \Phi(\kappa_2, \mu_2)$ *is not entanglement annihilating under the following conditions:*

(i) $\kappa_1 < 1$, $\kappa_2 < 1$,

$$a_1 < \frac{\kappa_1(1+a_2)}{2(1+a_2)-\kappa_2}, \qquad a_2 < \frac{\kappa_2(1+a_1)}{2(1+a_1)-\kappa_1};$$

(ii) $\kappa_1 < 1$, $\kappa_2 \geq 1$,

$$a_1 < \frac{\kappa_1(\kappa_2+a_2)}{\kappa_2+2a_2}, \qquad a_2 < 1 - \kappa_2 \frac{1+a_1-\kappa_1}{2(1+a_1)-\kappa_1};$$

(iii) $\kappa_1 \geq 1$, $\kappa_2 \geq 1$,

$$a_1 < 1 - \frac{\kappa_1 a_2}{\kappa_2 + 2a_2}, \qquad a_2 < 1 - \frac{\kappa_2 a_1}{\kappa_1 + 2a_1}.$$

Proof.
$\Phi[\varrho] = \pi^{-2} \iint d^2\alpha \, d^2\beta \, \tilde{A}_{\alpha\beta} \varrho \tilde{A}^{\dagger}_{\alpha\beta}$, where

$$\tilde{A}_{\alpha\beta} = \int \frac{d^2\gamma}{\pi\sqrt{\tau}} \exp\bigg(- \frac{|\alpha|^2 + |\beta|^2 + |\gamma|^2}{2} + \sqrt{1-\eta}\,\alpha\gamma + \frac{1}{2\tau} \left|\sqrt{\tau - 1}\,\beta + \sqrt{\eta}\,\gamma\right|^2 \bigg) \left|\sqrt{\tfrac{\tau-1}{\tau}}\,\beta + \sqrt{\tfrac{\eta}{\tau}}\,\gamma\right\rangle\langle\gamma|, \tag{1}$$

$$\eta = \frac{\kappa}{\tau}, \qquad \tau = \begin{cases} 1 + a, & 0 < \kappa < 1, \\ \kappa + a, & \kappa > 1. \end{cases} \tag{2}$$

(Follows from Ivan, Sabapathy, Simon'11)
The parameter η defines the attenuation factor of the quantum limited attenuation $\Phi_{\mathrm{QL}\eta}$ and τ defines the power gain of the quantum limited amplifier $\Phi_{\mathrm{QL}\tau}$, the concatenation of these channels results in the channel $\Phi(\kappa, \mu)$ given by (1), i.e.
$\Phi_{\mathrm{QL}\tau} \circ \Phi_{\mathrm{QL}\eta} = \Phi(\kappa, \mu)$

It suffices to find a two-mode state $|\psi\rangle$ such that $(\Phi(\kappa_1,\mu_1)\otimes\Phi(\kappa_2,\mu_2))\,[|\psi\rangle\langle\psi|]$ is entangled for parameters $\kappa_{1,2}$ and $a_{1,2}$ satisfying (i)–(iii). Let $|\psi\rangle = [2(1-e^{-|\gamma|^2})]^{-1/2}(|\gamma\rangle|0\rangle - |0\rangle|\gamma\rangle)$, its energy $\mathcal{E} = (1-e^{-|\gamma|^2})^{-1}|\gamma|^2 \to 1$ when $|\gamma|\to 0$.

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Entanglement witness:

$$W_\lambda = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} e^{\lambda(|\alpha|^2+|\beta|^2)}|\alpha\rangle\langle\beta|\otimes|\beta\rangle\langle\alpha|.$$

For all pure factorized states $|\xi\rangle\otimes|\upsilon\rangle$ we have $\mathrm{tr}[W_\lambda|\xi\rangle\langle\xi|\otimes|\upsilon\rangle\langle\upsilon|] = \left|\int\frac{d^2\alpha}{\pi}e^{\lambda|\alpha|^2}\langle\xi|\alpha\rangle\langle\alpha|\upsilon\rangle\right|^2 \geq 0$, whereas $\mathrm{tr}[W_\lambda\varrho] < 0$ indicates entanglement of ϱ.

$$
\begin{aligned}
&\mathrm{tr}\left\{W_\lambda\left(\Phi(\kappa_1,\mu_1)\otimes\Phi(\kappa_2,\mu_2)\right)[|\psi\rangle\langle\psi|]\right\}\\
&=\left[\left(1-e^{-|\gamma|^2}\right)\left(\tau_1\tau_2(1-\lambda)^2-(\tau_1-1)(\tau_2-1)\right)\right]^{-1}\\
&\times\left\{\exp\left[-\frac{\eta_1\tau_1\left(1-\lambda(2-\lambda)\tau_2\right)}{\tau_1\tau_2(1-\lambda)^2-(\tau_1-1)(\tau_2-1)}\,|\gamma|^2\right]\right.\\
&+\exp\left[-\frac{\eta_2\tau_2\left(1-\lambda(2-\lambda)\tau_1\right)}{\tau_1\tau_2(1-\lambda)^2-(\tau_1-1)(\tau_2-1)}\,|\gamma|^2\right]\\
&\left.-2\exp\left[-\left(1-\frac{\sqrt{\eta_1\tau_1\eta_2\tau_2}\,(1-\lambda)}{\tau_1\tau_2(1-\lambda)^2-(\tau_1-1)(\tau_2-1)}\right)|\gamma|^2\right]\right\},
\end{aligned}
$$

which is justified for $\lambda < \lambda_0 = 1 - \sqrt{(\tau_1-1)(\tau_2-1)/\tau_1\tau_2}$.

The average value obtained takes negative values in the widest region of parameters $\eta_{1,2}, \tau_{1,2}$ if $|\gamma| \to 0$. In this case, the output state is entangled if there exists a solution λ_* of the inequality $2[\tau_1\tau_2(1-\lambda)^2 - (\tau_1 - 1)(\tau_2 - 1) - \sqrt{\eta_1\tau_1\eta_2\tau_2}\,(1-\lambda)] <$ $\eta_1\tau_1 + \eta_2\tau_2 - \lambda(2-\lambda)\tau_1\tau_2(\eta_1 + \eta_2)$ such that $\lambda_* < \lambda_0$. The reader will have no difficulty in showing that such a solution λ_* exists if $2 - \eta_1 - \tau_2(2 - \eta_1 - \eta_2) > 0$ and $2 - \eta_2 - \tau_1(2 - \eta_1 - \eta_2) > 0$. Substituting expressions (2) for $\eta_{1,2}$ and $\tau_{1,2}$ yields formulas (i)–(iii).

2-local attenuator and amplifier: Non-gaussian inputs

$\kappa_1 = \kappa_2 = \kappa$, $\mu_1 = \mu_2 = \mu$: Φ is not 2-LEA if $\mu < \frac{1}{2}\sqrt{\kappa^2 + 1}$

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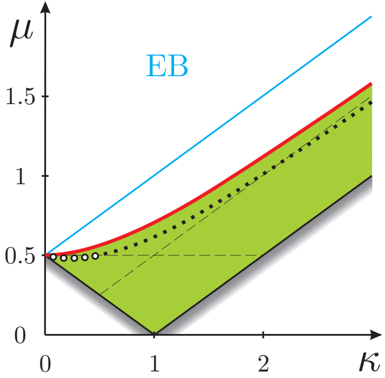

Dots: Sabapathy, Ivan, Simon'11 for the state $\frac{1}{\sqrt{2}}(|n\rangle|0\rangle + |0\rangle|n\rangle)$, $n = 5$

Proposition (SF, Ziman'14). The channel $\Phi(\kappa,\mu) \in \mathcal{C}$ is not N-LEA for any $N = 2, 3, \ldots$ if the total noise level satisfies $\mu < \frac{1}{2}\sqrt{\kappa^2 + 1}$.

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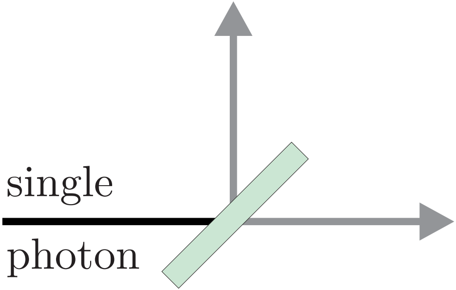
"Rather good" input state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle).$$

Proposition (SF, Ziman'14). The channel $\Phi(\kappa, \mu) \in \mathcal{C}$ is not N-LEA for any $N = 2, 3, \ldots$ if the total noise level satisfies $\mu < \frac{1}{2}\sqrt{\kappa^2 + 1}$.

"Rather good" input state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle).$$



Summary

- The channel $\Upsilon = \Phi(\kappa_1, \mu_1) \otimes \Phi(\kappa_2, \mu_2) \otimes \cdots \otimes \Phi(\kappa_N, \mu_N)$ annihilates entanglement of all gaussian N-mode states if $\mu_i \geq \frac{1}{2}$, $i = 1, \ldots, N$.
- Phase-insensitive amplification $\Phi_1 \otimes \Phi_2 \otimes \ldots \Phi_N$ with the power gain $\kappa_i \geq 2$ (≈ 3 dB, $i = 1, \ldots, N$) is shown to destroy entanglement of any N-mode gaussian state even in the case of quantum limited performance
- Special low-intensity non-Gaussian states remain entangled within a wide range of noises beyond quantum limited performance ($\mu \leq \frac{1}{2}\sqrt{\kappa^2 + 1}$) for any degree of attenuation or gain.

Thank you for listening!