UNCOMPUTABILITY AND COMPLEXITY OF QUANTUM CONTROL

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D.I. Bondar, A.N. Pechen, Uncomputability and complexity of quantum control, `arXiv:1907.10082`

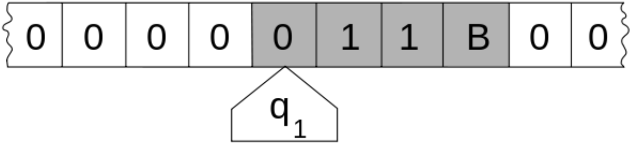
STRUCTURE

- (Un)computability
- Diophantine equations
- Quantum control: overview
- Digitizes quantum control
- Result

ALGORITHMS

Muhammad ibn Musa al-Khwarizmi
(780–850)

TURING MACHINE[1]



[1] A.M. Turing, On computable numbers, with an application to the Entscheidungsproblem, Proceedings of the London Mathematical Society, 2, 42 (1), pp. 230–65 (1936) (published 1937).)

HALTING PROBLEM AS AN UNDECIDABLE PROBLEM

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"Proof"

- $T_1, T_2, \ldots,$ – algorithms which have as input and output natural numbers.
- Suppose there exist algorithm $A(N, X)$:
  - halts and returns 1 if $T_N(X)$ does not halt
  - does not halt otherwise
- Then $D(N) = A(N, N)$ halts iff $T_N(N)$ does not halt.
- $D = T_K : D(K) = A(K, K)$ halts if $D(K)$ does not halt.

Contradiction!

DIOPHANTINE EQUATIONS

Diophantine equations are polynomial equations with *integer* coefficients and *integer* solutions:

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Examples:

- $x^2 - ny^2 = 1$ (Pell's eq.): $(1, 0)$ and $(x, y) : x/y \sim \sqrt{n}$
- $x^2 + y^2 = z^2$ (Pythagorean triples): (3,4,5), (5,12,13), ...
- $x^n + y^n = z^n$ for $n > 2$ (Fermat's Last Theorem): not solvable

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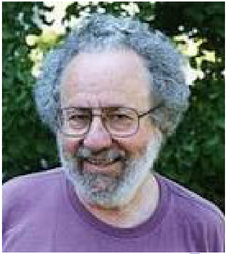
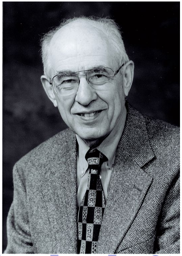
(Yuri Matiyasevich, Julia Robinson, Martin Davis, Hilary Putnam)





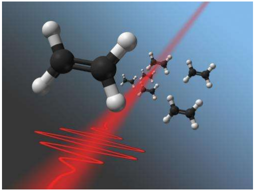
UNCOMPUTABLE PROBLEMS IN QUANTUM PHYSICS

Diophantine equation $D(x_1, \ldots, x_n) = 0$ has a solution iff n-boson's Hamiltonian $\hat{H} = D(a_1^+ a_1, \ldots, a_n^+ a_n)$ has the zero ground state. [T.D. Kieu, Int. J. Theor. Phys. 42, 1461 (2003).]

Hilbert's 10th problem$\longrightarrow$ The ground state of n-bosons is uncomputable [W. D. Smith, App. Math. Comp. 178, 184 (2006)]

Whether a quantum system is gapless is not decidable [T. S. Cubitt et al, Nature 528, 207 (2015)].

QUANTUM CONTROL OVERVIEW



$$H = H_0 + u(t)V, \qquad P_{i\to f}(u) \to \max$$

1980th: V. Belavkin, A. Butkovskiy, P. Brumer, H. Rabitz, S. Rice, Y. Samoilenko, M. Shapiro, D. Tannor, etc.

Now: Extremely high interest. Applications:

- Quantum computing
- Quantum information
- Laser-assisted chemistry, NMR.

Nobel Prize in Physics 2012: S. Haroche and D. Wineland for experimental manipulation of individual quantum systems.

TYPICAL CONTROL TASKS

State-to-state transfer:

$$\psi_\mathrm{i} \xrightarrow{\quad u(t) \quad} \psi_\mathrm{f}, \qquad P_{\mathrm{i}\to\mathrm{f}} = |\langle \psi_\mathrm{f}, U_T \psi_\mathrm{i} \rangle|^2 \to \max$$

$$\langle \hat{O}_T \rangle = \mathrm{Tr}\left[\hat{O} U_T \rho_\mathrm{i} U_T^\dagger \right] \to \max$$

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Quantum gate/process generation:

$$\mathbb{I} \xrightarrow{\quad u(t) \quad} W, \qquad |\mathrm{Tr} W^\dagger U_T| \to \max$$

Examples: $W = \mathrm{CNOT}, \mathrm{SWAP}, \mathrm{Toffoli}, \ldots$

QUANTUM CONTROL: STANDARD FORMULATION

Dynamics:

$$\frac{d\rho_t^u}{dt} = -i[H_0 + u(t)V, \rho_t^u] + \mathcal{L}(\rho_t^u), \qquad \rho_{t=0}^u = \rho_0$$

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Objective functional:

$$\mathscr{F}_O(u) \;=\; \mathrm{Tr}[O\rho_T^u] \to \max$$

COHERENT CONTROL AND CONTROLLABILITY

Coherent control

$$\frac{d|\psi_t\rangle}{dt} = -\frac{i}{\hbar}\Big(H_0 + u(t)V\Big)|\psi_t\rangle$$

Controllability

$$|\psi_\mathrm{i}\rangle \xrightarrow{u(t)} |\psi_\mathrm{f}\rangle = U_T|\psi_\mathrm{i}\rangle$$

Controllability criteria:

$$\mathrm{Lie}\{-iH_0, -iV\} \sim \mathfrak{su}(n)\ (\text{or } \mathfrak{sp}(n/2))$$

KRAUS MAPS

Kraus map Φ — most general transformation of density matrix:

- Linear map $\Phi : \rho \to \rho_T$;
- Completely positive, i.e., $\Phi \otimes \mathbb{I}_K \geq 0$;
- Trace preserving, i.e., $\mathrm{Tr}\Phi(\rho) = \mathrm{Tr}\rho$.

[2] R. Wu, A. Pechen, C. Brif, H. Rabitz, "Controllability of open quantum systems with Kraus-map dynamics", J. Phys. A: Math. Theor., **40**, 5681–5693 (2007).

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$$\Phi(\rho) = \sum_k K_k \rho K_k^+, \qquad \sum_k K_k^+ K_k = \mathbb{I}.$$

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For any target state ρ_{f}, there exist a universally optimal Kraus map $\Phi_{\rho_{\mathrm{f}}}$ such that for any ρ: $\Phi_{\rho_{\mathrm{f}}}(\rho) = \rho_{\mathrm{f}}$.[2]

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INCOHERENT CONTROL

General method for coherent and incoherent control of open quantum systems was developed[3]:

$$\frac{d\rho_t}{dt} = -\frac{i}{\hbar}[H_{u(t)}, \rho_t] + \mathcal{L}_{n(t)}(\rho_t),$$

Here u is coherent (e.g., laser field) and n is incoherent (e.g., temperature distribution) control.

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Kraus map:

$$\rho_0 \to \rho_T = \Phi_{(u,n)}(\rho_0) = \sum_k K_k \rho_0 K_k^+.$$

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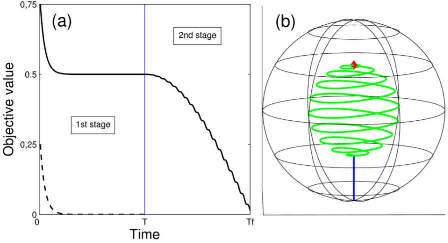
COMPLETE CONTROLLABILITY

Our general approach with coherent and incoherent controls was used to show[4] that for a generic N-level quantum system (for any N), there exist coherent u and incoherent n controls, which approximately steer **any initial state to any predefined target state for any N-level system**. In other words, open quantum systems are **approximately controllable in the set of all density matrices.** Allows to construct universally optimal Kraus maps.

An example with two levels of calcium atom: moving from the point $(0, 0, -1)$ to the point $(0, 0, 0.5)$ in the Bloch ball. Two stages of the control:
1) using only incoherent control;
2) using only coherent control.



https://arxiv.org/abs/1210.2281

[4] A. Pechen, "Engineering arbitrary pure and mixed quantum states", Phys. Rev. A. **84**, 042106 (2011). (arXiv:1210.2281 [quant-ph])

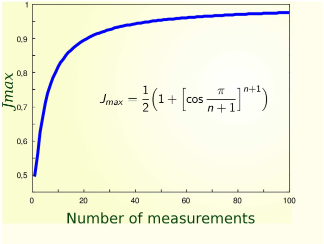
CONTROL BY QUANTUM MEASUREMENTS

- Belavkin (1983): Theory of controlling quantum systems measured in discrete or continuous time; Wiseman, Milburn (1990): Feedback control. Balachandran, Roy (2000): Quantum anti-Zeno effect.
- Pechen, Ilyn, Shuang, Rabitz (2006):[5] General method

Optimal control by non-selective quantum measurements. Exact analytical solution for a qubit by n measurements:

$$Q_i = \{P_i\}, \quad \mathcal{M}_Q(\rho) = \sum_i P_i \rho P_i$$

$$J = \mathrm{Tr}[O\mathcal{M}_{Q_n} \dots \mathcal{M}_{Q_1}(\rho)]$$



Applied by[6]

[5] A. Pechen, N. Il'in, F. Shuang, H. Rabitz, "Quantum control by von Neumann measurements", Phys. Rev. A, **74**, 052102 (2006).

[6] M. S. Blok et al, Nature Physics **10**, 189 (2014); H. W. Wiseman, Quantum control: Squinting at quantum systems, Nature **470**, 178 (2011); etc.

DIGITIZED QUANTUM CONTROL

DISCRETE CONTROLS: In experiments we always have access to a finite number N of controls.

[7] I.V. Volovich, Number Theory as the Ultimate Physical Theory, p-Adic Numbers, Ultrametric Analysis and Applications. **2**, 77–87 (2010); B. Dragovich, A.Yu. Khrennikov, S.V. Kozyrev, I.V. Volovich, E.I. Zelenov, p-Adic Mathematical Physics: The First 30 Years, p-Adic Numbers Ultrametric Anal. Appl. 9 (2), 87–121 (2017).

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RATIONAL NUMBERS IN PHYSICS: All experimental and observational numerical data are rational numbers.[7] $(O, \rho_0, K_{i,j})$

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N elementary controls are most generally described by Kraus maps

$$\Phi_i(\rho) = \sum_k K_{i,k} \rho K_{i,k}^\dagger, \qquad \sum_k K_{i,k}^\dagger K_{i,k} = \mathbb{I}$$

DQC is to find a control policy (if it exists) $\mathbf{p} = (p_1, \ldots, p_L) \in AP$ (the set of accessible controls) such that

$$\mathcal{F}(\mathbf{p}) := \mathrm{Tr}[O \Phi_{p_L} \ldots \Phi_{p_1}(\rho_0)] = 0$$

[7] I.V. Volovich, Number Theory as the Ultimate Physical Theory, p-Adic Numbers, Ultrametric Analysis and Applications. **2**, 77–87 (2010); B. Dragovich, A.Yu. Khrennikov, S.V. Kozyrev, I.V. Volovich, E.I. Zelenov, p-Adic Mathematical Physics: The First 30 Years, p-Adic Numbers Ultrametric Anal. Appl. 9 (2), 87–121 (2017).

DQC TO DIOPHANTINE EQUATIONS

Define

$$\hat{\phi}_k(i) = \sum_{l=1}^{N} K_{l,k} \prod_{j=1, j\neq l}^{N} \frac{i-j}{l-j} \text{ such that } \hat{\phi}_k(i) = K_{i,k} \text{ for } 1 \leq i \leq N$$

Then

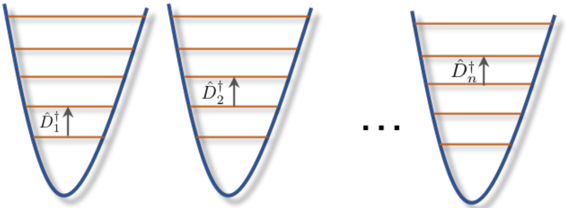
$$\mathcal{F}(\mathbf{p}) \rightarrow \mathscr{F}(\mathbf{p}) = \sum_{k_1,\ldots,k_P} \mathrm{Tr}\left(O \left[\prod_{l=L}^{1} \hat{\phi}_{k_l}(p_l) \right] \rho_0 \left[\prod_{m=L}^{1} \hat{\phi}_{k_m}(p_m) \right]^{\dagger} \right)$$

Solving the DQC is equivalent to solving the Diophantine equation:

$$\mathscr{F}^2(\mathbf{p}) + \prod_{\mathbf{p}' \in AC} \sum_{k=1}^{L} (p_k - p'_k)^2 = 0$$

DIOPHANTINE EQUATIONS TO DQC

$$D(x_1, \ldots, x_n) = 0 \Rightarrow \begin{cases} \rho_0 = |0, \ldots, 0\rangle\langle 0, \ldots, 0| \\ \Phi_i(\rho) = \hat{D}_i \rho \hat{D}_i^+, \qquad \hat{D}_i = \mathrm{e}^{a_i^+ - a_i} \\ \hat{O} = -D(a_1, \ldots, a_n)^+ D(a_1, \ldots, a_n) \end{cases}$$



Hilbert's 10th problem implies unsolvability of DQC!

EXAMPLE: NP-HARD TWO-BOSON PROBLEM

Deciding the solvability of the Diophantine equation

$$\alpha x_1^2 + \beta x_2 = \gamma$$

with respect to x_1 and x_2 is NP-hard.

Hence NP-hard to find a 2-mode state such that $\langle \hat{O} \rangle = 0$ with

$$\hat{O} = -(\alpha \hat{a}_1^+ \hat{a}_1^+ + \beta \hat{a}_2^+ - \gamma)(\alpha \hat{a}_1 \hat{a}_1 + \beta \hat{a}_2 - \gamma)$$

CONCLUSIONS

The negative answer to Hilbert's 10th problem implies that there is no algorithm deciding whether there is a control policy for connecting any given pair of quantum states or, more generally, maximizing expectation of an observable.

$$\psi_\mathrm{i} \xrightarrow{\quad \mathit{DQC?} \quad} \psi_\mathrm{f}$$

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